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Cayley–Klein Contractions of Quantum Orthogonal Groups in Cartesian Basis

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Abstract

Spaces of constant curvature and their motion groups are described most naturally in Cartesian basis. All these motion groups also known as CK groups are obtained from orthogonal group by contractions and analytical continuations. On the other hand quantum deformation of orthogonal group $SO(N)$ is most easily performed in so-called symplectic basis. We reformulate its standard quantum deformation to Cartesian basis and obtain all possible contractions of quantum orthogonal group $SO_q(N)$ both for untouched and transformed deformation parameter. It turned out, that similar to undeformed case all CK contractions of $SO_q(N)$ are realized. An algorithm for obtaining nonequivalent (as Hopf algebra) contracted quantum groups is suggested. Contractions of $SO_q(N),\ N=3,4,5$ are regarded as an examples.

1 Introduction

Systematic definitions of quantum deformations of classical simple Lie groups and algebras as well as descriptions their properties was given in [1]. Simple Lie groups and algebras are transformed by the contraction operation first introduced by E. Wigner and E. Inönü [2] to a nonsemisimple ones. Quantum analogues of the nonsemisimple low dimensions Lie algebras was obtained by contractions of quantum algebras $so_q(3), su_q(2)$, [3]–[7] and contractions of low dimensions quantum groups was discussed in [8]–[10]. Two types of contractions was discovered: with untouched deformation parameter (in [3],[6]

for quantum algebras and in [9],[10] for quantum groups) and with transformed deformation parameter [4],[5],[7],[8]. For the last case the quantum deformations of the algebras of the maximal symmetric motion groups of the $N-$dimensional flat spaces was constructed in [11]. $\gamma-$Poincare quantum group was obtained by contractions of the orthogonal quantum group $SO_q(N)$ [12]. Quantum Euclid group $E_\kappa(2)$ was described both by contraction of $SU_q(2)$ [13] and by direct quantization of Lie–Poisson structure [14]. A separate line of investigation is presented by the R-matrix approach to the quantum analogues of Euclid, Heisenberg and inhomogeneous groups [15]–[18].

It is well known [19] that the motion groups of all 3^{N-1} $(N-1)$-dimensional constant curvature spaces may be obtained by contractions and analytic continuations of the classical orthogonal group $SO(N)$. Cayley–Klein groups is the short name for this set of groups. The fundamental orthogonal $A^tA = I$ matrix $A \in SO(N)$ is replaced by the matrix $A(j)$ whose elements $(A(j))_{kp} = (k,p)a_{kp},\ (k,p) = \prod_{l=min\{k,p\}}^{max\{k,p\}-1} j_l,\ k,p = 1,\ldots,N$ are subject of the additional j-orthogonality relations $A(j)^tA(j) = 1$, where the parameters j_k takes three values each $j_k = 1, \iota_k, i$. The commutative $\iota_k\iota_p = \iota_p\iota_k \neq 0,\ k \neq p$ nilpotent $\iota_k^2 = 0$ units ι_k are corresponded to contractions and the imaginary unit $i^2 = -1$ to analytic continuations.

In the case of the quantum orthogonal group $SO_q(N)$ additionally the deformation parameter $q = \exp z$ is transformed as follows [20]: $z = Jv,\ J = (1,N)$, where v is the new deformation parameter . At the same time the quantum group contractions with untransformed deformation parameter are known [9],[10]. For unification of both such cases in one approach the concept of different couplings of Cayley-Klein and Hopf structures was suggested [21],[22]. It is well known that quantum groups are Hopf algebras and Cayley-Klein structure is defined by the distribution of the contraction parameters j among the elements of the generating matrix. For the quantum orthogonal group in so-called "symplectic" basis (where the invariant quadratic form for $q = 1$ is defined by the matrix C_0 with all null elements except units on the secondary diagonal) this concept was realized in [23]–[25] by the substitution in standard machinery of quantum group the generating matrix $T_\sigma(j) = D_\sigma A(j) D_\sigma^{-1},\ D_\sigma = DV_\sigma$, where the matrix D is the solution of the equation $D^tC_0D = I$ and describe transformation from Cartesian basis to symplectic one. The matrix $V_\sigma,\ (V_\sigma)_{ik} = \delta_{\sigma_i,k}$, where $\sigma \in S(N)$ is a N order permutation, define the distribution of the contraction parameters

in $T_\sigma(j)$. In this case the transformation of the deformation parameter depend on permutation σ. All permutations which leads to untouched $(J=1)$ deformation parameter and some permutations which correspond to transformed ones are enumerated in [23]–[25]. The contracted quantum groups $SO_v(N;j;\sigma)$ in these papers were regarded as Hopf algebra over Pimenov algebra $D(\iota)$ generated by nilpotent commutative generators. It turned out that not all Cayley–Klein contractions are admissible for quantum groups in this assumption which therefore is too restrictive.

The main statement of the algebraic structures contraction method is to take into account in all relations only principal parts with respect to tending to zero contraction parameter and to neglect all others. Therefore in this paper in all relations of quantum group theory only principal (complex) terms are taken into account and all other terms with nilpotent multipliers are neglected. Besides contractions of orthogonal quantum groups $SO_v(N;j;\sigma)$ are regarded in more usual Cartesian basis. For untouched deformation parameter results are the same as in [23]–[25] and for all other permutations deformation parameter is multiplied by $J=\bigcup_{k=1}^{n}(\sigma_k,\sigma_{k'})$, where n is integral part of $N/2$. The unification of multipliers $(\sigma_k,\sigma_p)\cup(\sigma_m,\sigma_r)$ is understood as the first power product of all parameters j_k which appear at least in one multiplier (σ_k,σ_p) or (σ_m,σ_r). For example, $(j_1j_2)\cup(j_2j_3)=j_1j_2j_3$. It turned out, that the full scheme of CK contractions are realized for the quantum group $SO_q(N)$. Not all identically contracted quantum groups corresponding to different permutations σ are nonisomorphic. Quantum groups isomorphism is connected with the notion of equivalent distributions of nilpotent parameters in generating matrix. Nonisomorphic contracted quantum groups are correspond in the first place to generating matrices with nonequivalent distributions of nilpotent parameters and secondly to equivalent generating matrices but with different transformations of deformation parameter $(J_1\neq J_2)$. As an example quantum groups $SO_v(3;j;\sigma)$ are considered in detail and nonisomorphic contractions are given for quantum groups $SO_v(N;j;\sigma),\ N=4,5$. The russian version of this paper was published in [26].

2 Definition of quantum group $SO_v(N;j;\sigma)$

Let us start with an algebra $\mathbf{D}\langle(U(j;\sigma))_{ik}\rangle$ of noncommutative polynomials of N^2 variables, which are an elements of generating matrix $(U(j;\sigma))_{ik}=$

$(\sigma_i,\sigma_k)u_{\sigma_i\sigma_k}$. Let us introduce the transformation of the deformation parameter $q=e^z$ as follows: $z=Jv$, where v is a new deformation parameter and J is some product of parameters j for the present unknown. Let $\tilde{R}_v(j),\tilde{C}_v(j)$ be matrices which are obtained from $\tilde{R}_q,\tilde{C}$ respectively by the replacement of deformation parameter z with Jv. The commutation relations of the generators $U(j;\sigma)$ are defined by

$$\tilde{R}_v(j)U_1(j;\sigma)U_2(j;\sigma)=U_2(j;\sigma)U_1(j;\sigma)\tilde{R}_v(j), \tag{1}$$

where

$$U_1(j;\sigma)=U(j;\sigma)\otimes I,\ U_2(j;\sigma)=I\otimes U(j;\sigma),$$
$$U(j;\sigma)=V_\sigma U(j)V_\sigma^{-1},\ (V_\sigma)_{ik}=\delta_{\sigma_i k},$$
$$\tilde{R}_v(j)=(D\otimes D)^{-1}R_v(j)(D\otimes D),\quad R_v(j)=R_q(z\rightarrow Jv),$$
$$D^{-1}=\frac{1}{\sqrt{2}}\left(\begin{array}{ccc} I & 0 & \tilde{C}_0 \\ 0 & \sqrt{2} & 0 \\ i\tilde{C}_0 & 0 & -iI \end{array}\right),\ N=2n+1,$$

$\tilde{C}_0$ is the $n\times n$ matrix with all null elements except units on the secondary diagonal and the explicit form of the matrix $\tilde{R}_q$ in Cartesian basis is given in Appendix 1. The additional relations of (v,j)-orthogonality are hold

$$U(j;\sigma)\tilde{C}_v(j)U^t(j;\sigma)=\tilde{C}_v(j),\ \ U^t(j;\sigma)\tilde{C}_v^{-1}(j)U(j;\sigma)=\tilde{C}_v^{-1}(j), \tag{2}$$

where $C=C_0q^\rho$, and $\rho=diag(\rho_1,\ldots,\rho_N),\ (C_0)_{ik}=\delta_{i'k},\ i,k=1,\ldots,N,\ i'=N+1-i,$ that is $(C)_{ik}=q^{\rho_{i'}}\delta_{i'k}$ and $(C^{-1})_{ik}=q^{-\rho_i}\delta_{i'k}$, $\tilde{C}_v(j)=D^{-1}C_v(j)(D^t)^{-1}$,

$$(\rho_1,\ldots,\rho_N)=\left\{\begin{array}{c}(n-\frac{1}{2},n-\frac{3}{2},\ldots,\frac{1}{2},0,-\frac{1}{2},\ldots,-n+\frac{1}{2}),\ N=2n+1\\ (n-1,n-2,\ldots,1,0,0,-1,\ldots,-n+1),\ N=2n.\end{array}\right.$$

The quantum orthogonal Cayley–Klein group $SO_v(N;j;\sigma)$ is defined as the quotient algebra of $\mathbf{D}\langle(U(j;\sigma))_{ik}\rangle$ by relations (1),(2).

Formally $SO_v(N;j;\sigma)$ is a Hopf algebra with the following coproduct Δ, counit ϵ and antipode S :

$$\epsilon(U(j;\sigma))=I,\ \ \Delta U(j;\sigma)=U(j;\sigma)\dot{\otimes}U(j;\sigma),$$
$$S(U(j;\sigma))=\tilde{C}_v(j)U^t(j;\sigma)\tilde{C}_v^{-1}(j), \tag{3}$$

where $(A\dot{\otimes}B)_{ik}=\sum_p A_{ip}\otimes B_{pk}$. The explicit form of antipode is given in Appendix 2 and $(v;j)$-orthogonality in Appendix 3.

Remark. All relations for the quantum group $SO_v(N;j;\sigma)$ may be obtained from the corresponding relations for $SO_q(N)$ in Cartesian basis [20] by replacement $z\rightarrow Jv$ and $u_{ik}\rightarrow(\sigma_i,\sigma_k)u_{\sigma_i\sigma_k}$.

3 The basic theorem

According to the algebraic structures contraction method in all relations of the previous section for nilpotent values of j only principal (complex) terms are taken into account and all other terms with nilpotent multipliers are neglected. Relation is called *admissible*, if it is possible to select a principal terms. Otherwise relation is called *inadmissible*. For example, equation $a + \iota_1 b + \iota_2 c = a_1 + \iota_1 d$ is admissible equation and is equivalent to $a = a_1$, whereas equation $\iota_1 b + \iota_2 c = \iota_1 \iota_2 d$ is inadmissible.

The formal definition of the quantum group $SO_v(N; j; \sigma)$ should be a real definition of contracted quantum group, if the proposed construction is a consistent Hopf algebra structure for the principal terms of all relations under nilpotent values of some or all parameters j. In other words, if all relations of the previous section are admissible. The following theorem holds.

To prove the admisibility of the relations it is necessary to have their explicit expressions. Such expressions are obtained for coproduct, counit, antipode and (v, j)-orthogonality relations for arbitrary N. Commutation relations (1) for generators of orthogonal quantum group are written as an overdetermined equation system and its explicit solution has obtained only for $N = 3$. The following theorem holds.

Theorem. *If the commutation relations for generators are defined and the deformation parameter is transformed as* $z = Jv,\ J = \bigcup_{k=1}^{n} (\sigma_k, \sigma_{k'})$, *then all Caley–Klein contractions of quantum groups* $SO_v(N; j; \sigma)$ *are allowed.*

Proof. Let us prove consistency of our construction for most singular case when all parameters j are nilpotent. Counit $\varepsilon(u_{\sigma_i \sigma_k}) = 0,\ i \neq k,\ \varepsilon(u_{\sigma_k \sigma_k}) = 1,\ k = 1, \ldots, n$ do not restrict the values of j. Multiplier $C_{ikr} = (\sigma_i, \sigma_r)(\sigma_r, \sigma_k)(\sigma_i, \sigma_k)^{-1}$ in coproduct $\Delta(u_{\sigma_i \sigma_k}) = \sum_{r=1}^{N} C_{ikr} u_{\sigma_i \sigma_r} \otimes u_{\sigma_r \sigma_k}$ is equal to 1, if $\sigma_i < \sigma_r < \sigma_k$, is equal to $(\sigma_k, \sigma_r)^2$, if $\sigma_i < \sigma_k < \sigma_r$, and is equal to $(\sigma_r, \sigma_k)^2$, if $\sigma_r < \sigma_k < \sigma_i$, therefore all expressions for coproduct are admissible for nilpotent values of all j. Because of symmetry $(\sigma_i, \sigma_k) = (\sigma_k, \sigma_i)$ it is sufficiently to examine the case $\sigma_k < \sigma_i$.

Let us analyze antipode $S(U(j; \sigma))$ (see Appendix 2). Terms $(\sigma_k, \sigma_{k'})^{-1} \sinh(Jv\rho_k)$, $k = 1, \ldots, n$, are appeared in expressions (51) for $p = n + 1 - k$. They are well defined if one take multiplier J equal to the first power product of all parameters j_k which appear at least in one multiplier $(\sigma_k, \sigma_{k'})^{-1}$, $k = 1, \ldots, n$,

that is $J = \bigcup\limits_{k=1}^{n}(\sigma_k, \sigma_{k'})$. Let us verify that all expressions for antipode are admissible. For nilpotent J one have $\sinh J = J$, $\cosh J = 1$. Two types of multipliers are appeared in antipode:

$$A_{kM}(\alpha) = J\left(\frac{(\sigma_k, \sigma_M)}{(\sigma_{k'}, \sigma_M)}\right)^{\alpha}, \quad B_{kM}(\alpha) = J^2\left(\frac{(\sigma_k, \sigma_M)}{(\sigma_{k'}, \sigma_{M'})}\right)^{\alpha},$$

where $k = 1, \ldots, n,\ M = 1, \ldots, N,\ \alpha = \pm 1$. All these multipliers are well defined for nilpotent values of j. Since $\alpha = \pm 1$, then without loss of generality one assume $\sigma_k < \sigma_{k'}$. For $A_{kM}(\alpha)$ there are three possibilities: (i) $\sigma_k < \sigma_M < \sigma_{k'}$, (ii) $\sigma_M \leq \sigma_k < \sigma_{k'}$, (iii) $\sigma_k < \sigma_{k'} \leq \sigma_M$. In the case (i) $A_{kM}(1) = (\sigma_k, \sigma_M)^2,\ A_{kM}(-1) = (\sigma_M, \sigma_{k'})^2$, in the case (ii) $A_{kM}(1) = 1,\ A_{kM}(-1) = (\sigma_k, \sigma_{k'})^2$, in the case (iii), on the contrary, $A_{kM}(1) = (\sigma_k, \sigma_{k'})^2,\ A_{kM}(-1) = 1$. Multipliers $B_{kM}(\alpha)$ all the more are well defined in view of J^2. In particular, for most unfavorable case $\sigma_{M'} < \sigma_M < \sigma_k < \sigma_{k'}$ one have the fraction $(\sigma_k, \sigma_M)(\sigma_{k'}\sigma_{M'})^{-1} = (\sigma_M, \sigma_{M'})^{-1}(\sigma_k, \sigma_{k'})^{-1}$, but J^2 contain the multiplier $(\sigma_M, \sigma_{M'})(\sigma_k, \sigma_{k'})$, therefore $B_{kM}(1)$ remains nonsingular. If $J = 1$, then $A_{kM}(\alpha) = B_{kM}(\alpha) = 1$. Because of arbitrary choice of k and M multipliers $A_{kM}(\alpha)$ and $B_{kM}(\alpha)$ are well defined for all values of k and M.

Besides Hopf structure the (v, j)-orthogonality relations (52–60),(61) are imposed on generators of quantum group $SO_v(N; j; \sigma)$. Equations (56),(52) and (7) for $k = p$ evidently are admissible. Equations (54),(55) for $p = n + 1 - k$ after division of both parts on $(\sigma_k, \sigma_{k'})$ have terms with multipliers $C_{kMk'}$, which are equal to 1, if $\sigma_k < \sigma_M < \sigma_{k'}$ and are some product of j_k^2 otherwise. Therefore these equations are admissible The rest equations of (v, j)-orthogonality have terms with coefficients

$$A_{KPM} = \frac{(\sigma_K, \sigma_M)(\sigma_M, \sigma_P)}{(\sigma_K, \sigma_P)}, \quad B_{KPr} = J\frac{(\sigma_K, \sigma_r)(\sigma_P, \sigma_{r'})}{(\sigma_K, \sigma_P)},$$

where $K, P, M = 1, \ldots, N$, $r = 1, \ldots, n$. These coefficients are well defined for all nilpotent values of j. For A_{KPM} it is easily follow from the analysis of three possible cases: (i) $\sigma_K < \sigma_M < \sigma_P$, (ii) $\sigma_M < \sigma_K < \sigma_P$, (iii) $\sigma_K < \sigma_P < \sigma_M$. Moreover in the case (i) $A_{KPM} = 1$ and corresponding terms are complex. Nonsingularity of B_{KPr} follows from simple analysis of three possible cases: (a) $\sigma_K < \sigma_r < \sigma_P < \sigma_{r'}$, (b) $\sigma_K < \sigma_P < \sigma_r < \sigma_{r'}$, (c) $\sigma_K < \sigma_r < \sigma_{r'} < \sigma_P$.

Thus we conclude, that (v,j)-orthogonality relations are admissible for any permutations and for nilpotent values of any parameters, therefore they do not restrict contractions of quantum group.

4 Nonisomorphic contracted quantum groups

If all parameters $j_k = 1$, then the map $u_{ik} \to (\sigma_i, \sigma_k) u_{\sigma_i \sigma_k}$ is invertible and all quantum groups $SO_v(N;j;\sigma)$ for any $\sigma \in S_N$ are isomorphic as Hopf algebras. Nonisomorphic quantum groups may appear under contractions when all or some parameters j take nilpotent values. It is clear that nonisomorphic quantum groups are appear under contractions with different numbers of parameters. Contractions on the same parameters, but with different transformations of deformation parameter (with different J) naturally give in result nonisomorphic quantum groups. Isomorphic quantum groups may appear under contractions of $SO_v(N;j;\sigma)$ with different σ by equal numbers of parameters, when multiplier J include equal numbers of parameters (but not necessarily the same) or when $J = 1$. In our approach contractions of quantum groups (even on equal numbers of parameters) are distinguished by the distributions of nilpotent parameters j in generating matrix $U(j;\sigma)$. Really, all relations of quantum group theory (commutators, (v,j)-orthogonality, antipode, coproduct and counit) depend on permutation σ by means of generating matrix, while matrices $R_v(j), C_v(j)$ depend on σ via transformations of deformation parameter, that is via J. Isomorphism of contracted quantum orthogonal groups is described by the following theorem.

Theorem. *Quantum groups $SO_v(N;j;\sigma_1)$ and $SO_w(N;j;\sigma_2)$ are isomorphic, if the following relations for their generators holds:*

$$U(j;\sigma_1) = V_\sigma U(j;\sigma_2) V_\sigma^{-1}, \tag{4}$$

where matrix $V_\sigma,\ \sigma \in S_N$ satisfy

$$(V_\sigma \otimes V_\sigma)\tilde{R}_w(j)(V_\sigma \otimes V_\sigma)^{-1} = \tilde{R}_v(j), \quad V_\sigma \tilde{C}_w(j) V_\sigma^t = \tilde{C}_v(j) \tag{5}$$

for $w = \pm v$ and $J_1 = J_2$ with possible replacement j_k on $j_{N-k},\ k = 1, \dots, N-1$.

Proof. Commutation relations (1) of $SO_v(N;j;\sigma_1)$ after transformation (4) take the form

$$\tilde{R}_v(j)(V_\sigma \otimes V_\sigma) U_1(j;\sigma_2) U_2(j;\sigma_2)(V_\sigma \otimes V_\sigma)^{-1} =$$

$$(V_\sigma \otimes V_\sigma) U_2(j;\sigma_2) U_1(j;\sigma_2)(V_\sigma \otimes V_\sigma)^{-1}\tilde{R}_v(j)$$

or after left multiplying on $(V_\sigma \otimes V_\sigma)^{-1}$ and right multiplying on $V_\sigma \otimes V_\sigma$, in the form

$$(V_\sigma \otimes V_\sigma)^{-1}\tilde{R}_v(j)(V_\sigma \otimes V_\sigma) U_1(j;\sigma_2) U_2(j;\sigma_2) =$$
$$U_2(j;\sigma_2) U_1(j;\sigma_2)(V_\sigma \otimes V_\sigma)^{-1}\tilde{R}_v(j)(V_\sigma \otimes V_\sigma),$$

which give first equation in (5). Antipode (3) after transformation (4) take the form

$$V_\sigma S(U(j;\sigma_2))V_\sigma^{-1} = \tilde{C}_v(j)\left(V_\sigma^{-1}\right)^t U^t(j;\sigma_2)V_\sigma^t \tilde{C}_v^{-1}(j)$$

or

$$S(U(j;\sigma_2)) = V_\sigma^{-1}\tilde{C}_v(j)\left(V_\sigma^{-1}\right)^t U^t(j;\sigma_2)V_\sigma^t \tilde{C}_v^{-1}(j)V_\sigma.$$

The last equation is just antipode of $SO_v(N;j;\sigma_2)$, if take into account the second equation in (5). At last, (v,j)-orthogonality relations (2) after (4) take the form

$$V_\sigma U(j;\sigma_2)V_\sigma^{-1}\tilde{C}_v(j)\left(V_\sigma^{-1}\right)^t U^t(j;\sigma_2)V_\sigma^t = \tilde{C}_v(j)$$

or

$$U(j;\sigma_2)V_\sigma^{-1}\tilde{C}_v(j)\left(V_\sigma^{-1}\right)^t U^t(j;\sigma_2) = V_\sigma^{-1}\tilde{C}_v(j)\left(V_\sigma^t\right)^{-1},$$

which evidently is condition (5) for matrix $\tilde{C}_v(j)$.

As a consequence of theorem is the following algorithm of obtaining of nonisomorphic contracted quantum groups. One call two distributions of nilpotent parameters among elements of generating matrices $U(j;\sigma_1), U(j;\sigma_2)$ *equivalent*, if they are connected by two operations: 1) they pass in each other by the permutations of the same columns and rows of generating matrices, that is by (4); 2) matrices pass in each other by reflection relative secondary diagonal with possible simultaneous replacement of j_k with $j_{N-k},\ k=1,\ldots,N-1$. Nonisomorphic contracted quantum groups are corresponded in the first place to the nonequivalent generating matrices and secondly to equivalent generating matrices, but with different transformations of deformation parameters ($J_1 \neq J_2$). For illustration of algorithm all nonequivalent contractions of quantum groups $SO_v(N;j;\sigma),\ N=3,4,5$ shall be regarded in the next sections.

5 Quantum groups $SO_v(3;j;\sigma)$

Quantum group $SO_q(3)$ has four nonisomorphic contracted groups: two Euclid groups $E^0_v(2)=SO_v(3;\iota_1,j_2;\sigma_0),\ J=\iota_1,\ E_z(2)=SO_z(3;\iota_1,1;\sigma),\ J=1$, where $\sigma_0=(1,2,3),\ \ \sigma=(2,1,3)$, and two Galilei groups $G^0_v(2)=SO_v(3;\iota_1,\iota_2;\sigma_0),\ \ J=\iota_1\iota_2,\ G_v(2)=SO_v(3;\iota_1,\iota_2;\sigma),\ J=\iota_2$. For comparison, nondeformed complex rotation group $SO(3)$ has two nonisomorphic Cayley–Klein contracted groups: Euclid group $E(2)$ and Galilei group $G(2)$.

5.1 Quantum groups $SO_v(3;j;\sigma_0),\ \sigma_0=(1,2,3)$

Let $C_1=\cosh Jv,\ S_1=\sinh Jv,\ J=j_1j_2$. Generating matrix

$$U(j)=\left(\begin{array}{ccc} u_{11} & j_1u_{12} & j_1j_2u_{13}\\ j_1u_{21} & u_{22} & j_2u_{23}\\ j_1j_2u_{31} & j_2u_{32} & u_{33}\end{array}\right) \tag{6}$$

satisfy (v,j)-orthogonality relations: (i) $U(j)C_v(j)U^t(j)=C_v(j)$, i.e.

$$iJS_1[u_{13},u_{11}]=C_1(u_{11}^2+J^2u_{13}^2-1)+j_1^2u_{12}^2,$$

$$iJS_1[u_{23},u_{21}]=C_1(j_1^2u_{21}^2+j_2^2u_{23}^2)+u_{22}^2-1,$$
$$iJS_1[u_{33},u_{31}]=C_1(J^2u_{31}^2+u_{33}^2-1)+j_2^2u_{32}^2,$$
$$u_{11}u_{21}j_1C_1-iu_{13}u_{21}j_1JS_1+j_1u_{12}u_{22}+u_{13}u_{23}j_2JC_1+iu_{11}u_{23}j_2S_1=0,$$
$$u_{11}u_{31}JC_1-iu_{13}u_{31}J^2S_1+Ju_{12}u_{32}+u_{13}u_{33}JC_1+iu_{11}u_{33}S_1=iS_1,$$
$$u_{21}u_{31}j_1JC_1-iu_{23}u_{31}j_2JS_1+j_2u_{22}u_{32}+j_2u_{23}u_{33}C_1+iu_{21}u_{33}j_1S_1=0,$$
$$u_{21}u_{11}j_1C_1-iu_{23}u_{11}j_2S_1+j_1u_{22}u_{12}+u_{23}u_{13}j_2JC_1+iu_{21}u_{13}j_1JS_1=0,$$
$$u_{31}u_{11}JC_1-iu_{33}u_{11}S_1+Ju_{32}u_{12}+u_{33}u_{13}JC_1+iu_{31}u_{13}J^2S_1=-iS_1,$$
$$u_{31}u_{21}j_1JC_1-iu_{33}u_{21}j_1S_1+j_2u_{32}u_{22}+u_{33}u_{23}j_2C_1+iu_{31}u_{23}j_2JS_1=0 \tag{7}$$

and (ii) $U(j)^tC_v^{-1}(j)U(j)=C_v^{-1}(j)$, i.e.

$$iJS_1[u_{11},u_{31}]=C_1(u_{11}^2+J^2u_{31}^2-1)+j_1^2u_{21}^2,$$

$$iJS_1[u_{12},u_{32}]=C_1(j_1^2u_{12}^2+j_2^2u_{32}^2)+u_{22}^2-1,$$
$$iJS_1[u_{13},u_{33}]=C_1(u_{33}^2+J^2u_{13}^2-1)+j_2^2u_{23}^2,$$

$$j_1u_{11}u_{12}C_1 + iu_{31}u_{12}j_1JS_1 + j_1u_{21}u_{22} + Ju_{31}u_{33}C_1 - iu_{11}u_{33}S_1 = 0,$$
$$Ju_{11}u_{13}C_1 + iu_{31}u_{13}J^2S_1 + Ju_{21}u_{23} + Ju_{13}u_{33}C_1 - iu_{11}u_{33}S_1 = -iS_1,$$
$$j_1Ju_{12}u_{13}C_1 + iu_{32}u_{13}j_2JS_1 + j_2u_{22}u_{23} + j_2u_{32}u_{33}C_1 - iu_{12}u_{33}j_1S_1 = 0,$$
$$j_1u_{12}u_{11}C_1 + iu_{32}u_{11}j_2S_1 + j_1u_{22}u_{21} + j_2Ju_{32}u_{31}C_1 - iu_{12}u_{31}j_1JS_1 = 0,$$
$$Ju_{13}u_{11}C_1 + iu_{33}u_{11}S_1 + Ju_{23}u_{21} + Ju_{33}u_{31}C_1 - iu_{13}u_{31}J^2S_1 = iS_1,$$
$$j_1Ju_{13}u_{12}C_1 + iu_{33}u_{12}j_1S_1 + j_2u_{23}u_{22} + j_2u_{33}u_{32}C_1 - iu_{13}u_{32}j_2JS_1 = 0. \quad (8)$$

There are three independent generators, for example, u_{12}, u_{13}, u_{23}, which are situated above diagonal. Their commutators are obtained from RUU-relations $\tilde{R}_v(j)U_1(j)U_2(j) = U_2(j)U_1(j)\tilde{R}_v(j)$ and are in the form

$$[u_{12}, u_{23}] = i\frac{\sinh Jv}{J}u_{22}(u_{11} - u_{33}),$$
$$[u_{13}, u_{23}] = u_{23}\left\{(\cosh Jv - 1)u_{13} - i\frac{\sinh Jv}{J}u_{33}\right\},$$
$$[u_{12}, u_{13}] = \left\{(\cosh Jv - 1)u_{13} + i\frac{\sinh Jv}{J}u_{11}\right\}u_{12}. \quad (9)$$

An associative algebra $SO_v(3; j; \sigma_0)$ is Hopf algebra with counit $\epsilon(U(j)) = I$, i.e. $\epsilon(u_{ik}) = 0, \epsilon(u_{kk}) = 1$, coproduct $\Delta U(j) = U(j) \otimes U(j)$ in the form

$$\Delta u_{12} = u_{11}\otimes u_{12} + u_{12}\otimes u_{22} + j_2^2u_{13}\otimes u_{32},\ \Delta u_{21} = u_{21}\otimes u_{11} + u_{22}\otimes u_{21} + j_2^2u_{23}\otimes u_{31},$$
$$\Delta u_{23} = u_{22}\otimes u_{23} + u_{23}\otimes u_{33} + j_1^2u_{21}\otimes u_{13},\ \Delta u_{32} = u_{32}\otimes u_{22} + u_{33}\otimes u_{32} + j_1^2u_{31}\otimes u_{12},$$
$$\Delta u_{13} = u_{11}\otimes u_{13} + u_{12}\otimes u_{23} + u_{13}\otimes u_{33},\ \Delta u_{31} = u_{31}\otimes u_{11} + u_{32}\otimes u_{21} + u_{33}\otimes u_{31},$$
$$\Delta u_{11} = u_{11} \otimes u_{11} + j_1^2u_{12} \otimes u_{21} + J^2u_{13} \otimes u_{31},$$
$$\Delta u_{22} = u_{22} \otimes u_{22} + j_1^2u_{21} \otimes u_{12} + j_2^2u_{23} \otimes u_{32},$$
$$\Delta u_{33} = u_{33} \otimes u_{33} + j_2^2u_{32} \otimes u_{23} + J^2u_{31} \otimes u_{13}, \quad (10)$$

and antipode $S(u(j)) = C_v(j)U^t(j)C_v^{-1}(j)$, where

$$S(u_{12}) = u_{21}\cosh(\frac{Jv}{2}) + ij_2^2u_{23}\frac{1}{J}\sinh(\frac{Jv}{2}),$$
$$S(u_{21}) = u_{12}\cosh(\frac{Jv}{2}) + ij_2^2u_{32}\frac{1}{J}\sinh(\frac{Jv}{2}),$$

$$S(u_{23}) = u_{32}\cosh(\frac{Jv}{2}) - ij_1^2 u_{12}\frac{1}{J}\sinh(\frac{Jv}{2}),$$

$$S(u_{32}) = u_{23}\cosh(\frac{Jv}{2}) - ij_1^2 u_{21}\frac{1}{J}\sinh(\frac{Jv}{2}),$$

$$S(u_{13}) = u_{31}\cosh^2(\frac{Jv}{2}) + u_{13}\sinh^2(\frac{Jv}{2}) + i\frac{1}{2}(u_{33}-u_{11})\frac{1}{J}\sinh(Jv),$$

$$S(u_{31}) = u_{13}\cosh^2(\frac{Jv}{2}) + u_{31}\sinh^2(\frac{Jv}{2}) + i\frac{1}{2}(u_{33}-u_{11})\frac{1}{J}\sinh(Jv),$$

$$S(u_{11}) = u_{11}\cosh^2(\frac{Jv}{2}) - u_{33}\sinh^2(\frac{Jv}{2}) + i\frac{1}{2}(u_{13}+u_{31})J\sinh(Jv),$$

$$S(u_{33}) = u_{33}\cosh^2(\frac{Jv}{2}) - u_{11}\sinh^2(\frac{Jv}{2}) - i\frac{1}{2}(u_{13}+u_{31})J\sinh(Jv),$$

$$S(u_{22}) = u_{22}. \tag{11}$$

Remark. Coproduct and counit of $SO_v(3;j;\sigma)$ are the same for any permutation σ. Only antipode, commutation and $(v;j)$-orthogonality relations are depend on σ.

For $j_1 = \iota_1$ **quantum Euclid group** $E_v^0(2) = SO_v(3;\iota_1,j_2;\sigma_0),\ J=\iota_1$ is obtained. From $(v;j)$-orthogonality relations it follows $u_{11}=1,\ u_{22} = u_{33},\ u_{23} = -u_{32}$, and from RUU-equations it follows that all these generators commute and generate rotation group $SO(2)$. Therefore it is naturally to introduce new notations $u_{22} = u_{33} = \cos\varphi,\ u_{23} = \sin\varphi = -u_{32}$, and rewrite generating matrix as

$$U(\iota_1;\sigma_0) = \begin{pmatrix} 1 & \iota_1 u_{12} & \iota_1 u_{13} \\ \iota_1 u_{21} & \cos\varphi & \sin\varphi \\ \iota_1 u_{31} & -\sin\varphi & \cos\varphi \end{pmatrix} \sim \begin{pmatrix} \cdot & \circ & \circ \\ & \cdot & \cdot \\ & & \cdot \end{pmatrix}, \tag{12}$$

where from $(v;j)$-orthogonality relations it follows

$$u_{21} = -(u_{12}\cos\varphi + u_{13}\sin\varphi + i\frac{v}{2}\sin\varphi),$$

$$u_{31} = u_{12}\sin\varphi - u_{13}\cos\varphi + i\frac{v}{2}(1-\cos\varphi). \tag{13}$$

Here and later distribution of nilpotent parameters among elements of generating matrix is shown with the help of notations: $\circ = \iota_1,\ \bullet = \iota_2,\ \times = \iota_1\iota_2$. (Let us remind that this distribution is symmetric relatively diagonal). Dots

denote complex elements. Commutation relations of independent generators are as follows

$$[u_{12}, \sin\varphi] = iv\cos\varphi(1-\cos\varphi),$$
$$[\sin\varphi, u_{13}] = iv\sin\varphi\cos\varphi, \quad [u_{12}, u_{13}] = ivu_{12}. \tag{14}$$

Coproduct of quantum Euclid group is given by

$$\Delta u_{12} = 1\otimes u_{12} + u_{12}\otimes\cos\varphi - j_2^2 u_{13}\otimes\sin\varphi,$$
$$\Delta u_{13} = 1\otimes u_{13} + u_{12}\otimes\sin\varphi + u_{13}\otimes\cos\varphi,$$
$$\Delta\sin\varphi = \cos\varphi\otimes\sin\varphi + \sin\varphi\otimes\cos\varphi, \quad \Delta\varphi = 1\otimes\varphi + \varphi\otimes 1, \tag{15}$$

antipode is as follows

$$S(u_{12}) = -u_{12}\cos\varphi - u_{13}\sin\varphi, \quad S(u_{13}) = -u_{13}\cos\varphi + u_{12}\sin\varphi,$$
$$S(\varphi) = -\varphi, \tag{16}$$

and their counit is equal to zero: $\epsilon(u_{12}) = \epsilon(\varphi) = \epsilon(u_{13}) = 0$.

If $u_{21},\ u_{31},\ \varphi$ are taken as independent generators, then equations (13)–(16) are rewritten in the following way: from $(v;j)$-orthogonality relations

$$u_{12} = -u_{21}\cos\varphi + u_{31}\sin\varphi - i\frac{v}{2}\sin\varphi,$$
$$u_{13} = -u_{21}\sin\varphi - u_{31}\cos\varphi - i\frac{v}{2}(1-\cos\varphi), \tag{17}$$

commutation relations

$$[u_{21}, \sin\varphi] = iv\cos\varphi(1-\cos\varphi),$$
$$[\sin\varphi, u_{31}] = -iv\sin\varphi\cos\varphi, \quad [u_{31}, u_{21}] = ivu_{21}, \tag{18}$$

coproduct

$$\Delta u_{21} = u_{21}\otimes 1 + \cos\varphi\otimes u_{21} + \sin\varphi\otimes u_{31},$$
$$\Delta u_{31} = u_{31}\otimes 1 - \sin\varphi\otimes u_{21} + \cos\varphi\otimes u_{31},$$
$$\Delta\varphi = 1\otimes\varphi + \varphi\otimes 1, \tag{19}$$

antipode

$$S(u_{21}) = -u_{21}\cos\varphi + u_{31}\sin\varphi - iv\sin\varphi,$$
$$S(u_{31}) = -u_{31}\cos\varphi - u_{21}\sin\varphi + iv(\cos\varphi - 1), \quad S(\varphi) = -\varphi \tag{20}$$

and counit $\epsilon(u_{21}) = \epsilon(\varphi) = \epsilon(u_{31}) = 0$.

Under contraction $j_2 = \iota_2$ **quantum analog** $N_v^0(2) = SO_v(3; j_1, \iota_2; \sigma_0),\ J = \iota_2$ **of cylindrical group or Newton group** $N(2)$ is obtained. Similarly to previous case with the help of (v, j)-orthogonality relations the generating matrix may be written in the form

$$U(\iota_2; \sigma_0) = \begin{pmatrix} \cos\psi & \sin\psi & \iota_2 u_{13} \\ -\sin\psi & \cos\psi & \iota_2 u_{23} \\ \iota_2 u_{31} & \iota_2 u_{32} & 1 \end{pmatrix} \sim \begin{pmatrix} \cdot & \cdot & \bullet \\ & \cdot & \bullet \\ & & \cdot \end{pmatrix}, \tag{21}$$

where

$$u_{31} = u_{23}\sin\psi - u_{13}\cos\psi + i\frac{v}{2}(1 - \cos\psi),$$
$$u_{32} = -u_{23}\cos\psi - u_{13}\sin\psi - i\frac{v}{2}\sin\psi, \tag{22}$$

and independent generators are subject of commutation relations

$$[\sin\psi, u_{23}] = iv\cos\psi(\cos\psi - 1),$$
$$[u_{23}, u_{13}] = ivu_{23}, \quad [\sin\psi, u_{13}] = iv\sin\psi\cos\psi. \tag{23}$$

Hopf algebra is defined by coproduct

$$\Delta(\sin\psi) = \cos\psi \otimes \sin\psi + \sin\psi \otimes \cos\psi, \quad \Delta(\psi) = 1 \otimes \psi + \psi \otimes 1,$$

$$\Delta u_{13} = u_{13} \otimes 1 + \cos\psi \otimes u_{13} + \sin\psi \otimes u_{23},$$
$$\Delta u_{23} = u_{23} \otimes 1 + \cos\psi \otimes u_{23} - j_1^2 \sin\psi \otimes u_{13}, \tag{24}$$

by antipode

$$S(u_{13}) = u_{31} + i\frac{v}{2}(u_{33} - u_{11}) = u_{23}\sin\psi - u_{13}\cos\psi + iv(1 - \cos\psi),$$

$$S(u_{23}) = u_{32} - i\frac{v}{2}j_1^2 u_{12} = -u_{23}\cos\psi - u_{13}\sin\psi - iv\sin\psi, \quad S(\psi) = -\psi, \tag{25}$$

and by counit $\epsilon(\psi) = \epsilon(u_{13}) = \epsilon(u_{23}) = 0$.

The distribution of ι_1 in matrix (12) is passed to the distribution of ι_2 in matrix (21) under reflection on secondary diagonal and simultaneous substitution $J = \iota_1$ by $J = \iota_2$. This means that the quantum Euclid group $E_v^0(2) = SO_v(3; \iota_1, 1; \sigma_0)$ is isomorphic to the quantum Newton group $N_v^0(2) = SO_v(3; 1, \iota_2; \sigma_0)$ as well as in nondeformed case. Under substitution

u_{31} on u_{13}, u_{21} on u_{23}, φ on $-\psi$, v on $-v$ commutation relations (18) are transformed in (23), coproduct (19) is transformed in (24) and antipode — in (25).

Two-dimensional contraction $j_1 = \iota_1$, $j_2 = \iota_2$ gives **quantum Galilei group** $G_v^0(2) = SO_v(3;\iota_1,\iota_2;\sigma_0),\ J=\iota_1\iota_2$. With the help of $(v;j)$-orthogonality relations the generating matrix may be written in the form

$$U(\iota;\sigma_0) = \left(\begin{array}{ccc} 1 & \iota_1 u_{12} & \iota_1\iota_2 u_{13} \\ -\iota_1 u_{12} & 1 & \iota_2 u_{23} \\ \iota_1\iota_2 u_{31} & -\iota_2 u_{23} & 1 \end{array}\right) \sim \left(\begin{array}{ccc} \cdot & \circ & \times \\ & \cdot & \bullet \\ & & \cdot \end{array}\right), \tag{26}$$

where $u_{31} = -u_{13} + u_{12}u_{23}$, and independent generators satisfy commutation relations

$$[u_{12},u_{23}] = 0, \quad [u_{23},u_{13}] = ivu_{23}, \quad [u_{12},u_{13}] = ivu_{12}. \tag{27}$$

Hopf algebra structure is given by coproduct

$$\Delta u_{12} = 1\otimes u_{12} + u_{12}\otimes 1, \quad \Delta u_{23} = 1\otimes u_{23} + u_{23}\otimes 1,$$
$$\Delta u_{13} = 1\otimes u_{13} + u_{13}\otimes 1 + u_{12}\otimes u_{23}, \tag{28}$$

antipode

$$S(u_{12}) = -u_{12}, \quad S(u_{13}) = -u_{13} + u_{12}u_{23}, \quad S(u_{23}) = -u_{23} \tag{29}$$

and standard counit $\epsilon(u_{12}) = \epsilon(u_{13}) = \epsilon(u_{23}) = 0$.

5.2 Quantum groups $SO_v(3;j;\sigma),\ \sigma=(2,1,3)$

Deformation parameter is transformed by multiplication on $J=(\sigma_1,\sigma_3) = (2,3) = j_2$. Commutators, (v,j)-orthogonality relations and antipode are easily obtained from corresponding formulas of $SO_z(3) = SO_v(3;j=1;\sigma_0)$ by interchange of indices 1 and 2 and then by standard reconstruction of contraction parameters j. In particular, generating matrix is as follows

$$U(j;\sigma) = \left(\begin{array}{ccc} u_{22} & j_1 u_{21} & j_2 u_{23} \\ j_1 u_{12} & u_{11} & j_1 j_2 u_{13} \\ j_2 u_{32} & j_1 j_2 u_{31} & u_{33} \end{array}\right), \tag{30}$$

Commutation relations of independent generators are

$$j_1^2[u_{21},u_{13}] = i\frac{1}{j_2}\sinh(j_2 v)u_{11}(u_{22}-u_{33}),$$

$$[u_{23}, u_{13}] = u_{13}\left\{\frac{1}{j_2}(\cosh j_2 v - 1)u_{23} - i\frac{1}{j_2}\sinh(j_2 v)u_{33}\right\},$$

$$[u_{21}, u_{23}] = \left\{\frac{1}{j_2}(\cosh j_2 v - 1)u_{23} + i\frac{1}{j_2}\sinh(j_2 v)u_{22}\right\}u_{21}. \quad (31)$$

Antipode is easily obtained by the transformations of (11)

$$S(u_{21}) = u_{12}\cosh(j_2\frac{v}{2}) + ij_2^2 u_{13}\frac{1}{j_2}\sinh(j_2\frac{v}{2}),$$

$$S(u_{12}) = u_{21}\cosh(j_2\frac{v}{2}) + ij_2^2 u_{31}\frac{1}{j_2}\sinh(j_2\frac{v}{2}),$$

$$S(u_{13}) = u_{31}\cosh(j_2\frac{v}{2}) - iu_{21}\frac{1}{j_2}\sinh(j_2\frac{v}{2}),$$

$$S(u_{31}) = u_{13}\cosh(j_2\frac{v}{2}) - iu_{12}\frac{1}{j_2}\sinh(j_2\frac{v}{2}),$$

$$S(u_{23}) = u_{32}\cosh^2(j_2\frac{v}{2}) + u_{23}\sinh^2(j_2\frac{v}{2}) + i\frac{1}{2}(u_{33} - u_{22})\frac{1}{j_2}\sinh(j_2 v),$$

$$S(u_{32}) = u_{23}\cosh^2(j_2\frac{v}{2}) + u_{32}\sinh^2(j_2\frac{v}{2}) + i\frac{1}{2}(u_{33} - u_{22})\frac{1}{j_2}\sinh(j_2 v),$$

$$S(u_{22}) = u_{22}\cosh^2(j_2\frac{v}{2}) - u_{33}\sinh^2(j_2\frac{v}{2}) + \frac{i}{2}(u_{23} + u_{32})j_2\sinh(j_2 v),$$

$$S(u_{33}) = u_{33}\cosh^2(j_2\frac{v}{2}) - u_{22}\sinh^2(j_2\frac{v}{2}) - \frac{i}{2}(u_{23} + u_{32})j_2\sinh(j_2 v),$$

$$S(u_{11}) = u_{11}. \quad (32)$$

Coproduct and counit are not changed and are given by (10), which correspond to identical permutation σ_0.

Contraction $j_1 = \iota_1$ left deformation parameter fixed since $J = j_2 = 1$ and gives **new quantum Euclid group** $E_z(2) = SO_z(3; \iota_1, 1; \sigma)$ with the matrix

$$U(\iota_1; \sigma) = \begin{pmatrix} \cos\varphi & \iota_1 u_{21} & \sin\varphi \\ \iota_1 u_{12} & 1 & \iota_1 u_{13} \\ -\sin\varphi & \iota_1 u_{31} & \cos\varphi \end{pmatrix} \sim \begin{pmatrix} \cdot & \circ & \cdot \\ & \cdot & \circ \\ & & \cdot \end{pmatrix}, \quad (33)$$

where generators are

$$u_{11} = 1,\ u_{22} = u_{33} = \cos\varphi,\ u_{23} = -u_{32} = \sin\varphi,$$

$$u_{12}\cos(\varphi - i\frac{v}{2}) = -(u_{21} + u_{13}\sin(\varphi - i\frac{v}{2})),$$
$$u_{31}\cos(\varphi - i\frac{v}{2}) = -(u_{13} + u_{21}\sin(\varphi - i\frac{v}{2})), \tag{34}$$

and the following commutation relations

$$[u_{21}, u_{13}] = 0, \quad [u_{13}, \sin\varphi] = 2i\mathrm{sinh}\frac{z}{2}u_{13}\cos(\varphi - i\frac{z}{2}),$$
$$[u_{21}, \sin\varphi] = 2i\mathrm{sinh}\frac{z}{2}\cos(\varphi + i\frac{z}{2})u_{21} \tag{35}$$

are holds. Antipode is given by

$$S(u_{21}) = u_{12}\mathrm{cosh}\frac{z}{2} + iu_{13}\mathrm{sinh}\frac{z}{2}, \quad S(u_{13}) = u_{31}\mathrm{cosh}\frac{z}{2} - iu_{21}\mathrm{sinh}\frac{z}{2},$$
$$S(\varphi) = -\varphi, \tag{36}$$

and coproduct is in the form

$$\Delta u_{13} = 1 \otimes u_{13} + u_{13} \otimes \cos\varphi + u_{12} \otimes \sin\varphi,$$
$$\Delta u_{21} = \cos\varphi \otimes u_{21} + u_{21} \otimes 1 + \sin\varphi \otimes u_{31}, \quad \Delta\varphi = 1 \otimes \varphi + \varphi \otimes 1. \tag{37}$$

Quantum Newton group $N_v(2) = SO_v(3; 1, \iota_2; \sigma)$, $J = \iota_2$ is described by relations $u_{33} = 1$, $u_{11} = u_{22} = \cos\psi$, $u_{21} = \sin\psi = -u_{12}$, i.e. the generating matrix is in the form

$$U(\iota_2; \sigma) = \begin{pmatrix} \cos\psi & \sin\psi & \iota_2 u_{23} \\ -\sin\psi & \cos\psi & \iota_2 u_{13} \\ \iota_2 u_{32} & \iota_2 u_{31} & 1 \end{pmatrix} \sim \begin{pmatrix} \cdot & \cdot & \bullet \\ & \cdot & \bullet \\ & & \cdot \end{pmatrix}, \tag{38}$$

where

$$u_{31} = -u_{13}\cos\psi - u_{23}\sin\psi - i\frac{v}{2}\sin\psi,$$
$$u_{32} = -u_{23}\cos\psi + u_{13}\sin\psi + i\frac{v}{2}(1 - \cos\psi), \tag{39}$$

and commutation relations

$$[\sin\psi, u_{13}] = iv\cos\psi(\cos\psi - 1),$$
$$[\sin\psi, u_{23}] = iv\cos\psi\sin\psi, \quad [u_{23}, u_{13}] = -ivu_{13} \tag{40}$$

are holds for independent generators. Antipode is given by

$$S(u_{13}) = -u_{13}\cos\psi - u_{23}\sin\psi - iv\sin\psi, \quad S(\psi) = -\psi,$$
$$S(u_{23}) = -u_{23}\cos\psi + u_{13}\sin\psi + iv(1-\cos\psi), \tag{41}$$

and coproduct is

$$\Delta\psi = 1\otimes\psi + \psi\otimes 1, \;\; \Delta u_{23} = u_{23}\otimes 1 + \cos\psi\otimes u_{23} + \sin\psi\otimes u_{13},$$
$$\Delta u_{13} = u_{13}\otimes 1 + \cos\psi\otimes u_{13} - \sin\psi\otimes u_{23}. \tag{42}$$

Generating matrices (38) and (21) are equal from the viewpoint of nilpotent units distribution, while formulae (39)–(42) pass to (22)–(25) under substitution u_{13} on u_{23} and u_{23} on u_{13}. Thus, both quantum groups are isomorphic $N_v(2) \simeq N_v^0(2) \simeq E_v^0(2)$.

For **quantum Galilei group** $G_v(2) = SO_v(3;\iota_1,\iota_2;\sigma),\; J=\iota_2$ it follows from (v,j)-orthogonality relations that $u_{11}=u_{22}=u_{33}=1$ and generating matrix takes the form

$$U(\iota;\sigma) = \left(\begin{array}{ccc} 1 & \iota_1 u_{21} & \iota_2 u_{23} \\ -\iota_1 u_{12} & 1 & \iota_1\iota_2 u_{13} \\ -\iota_2 u_{23} & \iota_1\iota_2 u_{31} & 1 \end{array}\right) \sim \left(\begin{array}{ccc} \cdot & \circ & \bullet \\ & \cdot & \times \\ & & \cdot \end{array}\right), \tag{43}$$

where $u_{31} = -u_{13} - u_{21}u_{23} + i\frac{v}{2}u_{21}$, commutation relations are

$$[u_{21},u_{13}] = 0, \quad [u_{23},u_{13}] = -ivu_{13}, \quad [u_{21},u_{23}] = ivu_{21}, \tag{44}$$

antipode may be written as

$$S(u_{21}) = -u_{21}, \quad S(u_{23}) = -u_{23}, \quad S(u_{13}) = -u_{13} - u_{21}u_{23}, \tag{45}$$

and coproduct is

$$\Delta u_{21} = 1\otimes u_{21} + u_{21}\otimes 1, \;\; \Delta u_{23} = 1\otimes u_{23} + u_{23}\otimes 1,$$
$$\Delta u_{13} = 1\otimes u_{13} + u_{13}\otimes 1 + u_{21}\otimes u_{23}. \tag{46}$$

Let us stress that $G_v(2)$ is not isomorphic to $G_v^0(2)$, in spite of the fact that both matrices (43), (26) are equivalent from the viewpoint of nilpotent units distribution, but deformation parameters are transformed in a different ways, namely, with multipliers $J=\iota_2$ and $J=\iota_1\iota_2$ respectively. Therefore commutation relations (27), (44), antipodes (29), (45) and counits are passed in each other under substitution u_{13} on u_{23} and vice versa, but in coproduct (28) $\Delta(u_{13})$ is not passed in $\Delta(u_{23})$ from (46).

5.3 Quantum groups $SO_v(3;j;\sigma),\ \sigma=(1,3,2)$

Deformation parameter is multiplied by $J=(\sigma_1,\sigma_3)=(1,2)=j_1$. Commutators, (v,j)-orthogonality relations and antipode are easily obtained from corresponding formulas of $SO_z(3)=SO_v(3;1,1;\sigma_0)$ by interchange of indices 2 and 3 and then by standard reconstruction of contraction parameters j. In particular, generating matrix is as follows

$$U(j;\sigma)=\left(\begin{array}{ccc} u_{11} & j_1j_2u_{13} & j_1u_{12} \\ j_1j_2u_{31} & u_{33} & j_2u_{32} \\ j_1u_{21} & j_2u_{23} & u_{22} \end{array}\right). \tag{47}$$

For $j_1=\iota_1$ **quantum Euclid group** $\tilde{E}_v(2)=SO_v(3;\iota_1,1;\sigma)$ is obtained with generators

$$U(\iota_1;\sigma)=\left(\begin{array}{ccc} 1 & \iota_1u_{13} & \iota_1u_{12} \\ \iota_1u_{31} & \cos\varphi & \sin\varphi \\ \iota_1u_{21} & -\sin\varphi & \cos\varphi \end{array}\right)\sim\left(\begin{array}{ccc} \cdot & \circ & \circ \\ & \cdot & \cdot \\ & & \cdot \end{array}\right). \tag{48}$$

As far as the generating matrix (48) is equal to (12), then $\tilde{E}_v(2)$ is isomorphic with $E_v^0(2)$ and therefore do not present a new quantum group.

Quantum Newton group $\tilde{N}_z(2)=SO_z(3;1,\iota_2;\sigma)$ is described by untouched deformation parameter z, generators $u_{33}=1,\ u_{11}=u_{22}=\cos\psi,\ u_{12}=\sin\psi=-u_{21}$, which are arranged in matrix form

$$U(\iota_2;\sigma)=\left(\begin{array}{ccc} \cos\psi & \iota_2u_{13} & \sin\psi \\ \iota_2u_{31} & 1 & \iota_2u_{32} \\ -\sin\psi & \iota_2u_{23} & \cos\psi \end{array}\right)\sim\left(\begin{array}{ccc} \cdot & \bullet & \cdot \\ & \cdot & \bullet \\ & & \cdot \end{array}\right). \tag{49}$$

This quantum group as Hopf algebra is isomorphic to quantum Euclid group $E_z(2)$ with untouched deformation parameter ($J=1$), since the generating matrix (49) is equal to (33), if instead of ι_2 put ι_1. Finally, **quantum Galilei group** $\tilde{G}_v(2)=SO_v(3;\iota_1,\iota_2;\sigma)$ is characterized by $J=\iota_1$, diagonal generators are equal to one $u_{11}=u_{22}=u_{33}=1$, and generating matrix is as follows

$$U(\iota;\sigma)=\left(\begin{array}{ccc} 1 & \iota_1\iota_2u_{13} & \iota_1u_{12} \\ \iota_1\iota_2u_{31} & 1 & \iota_2u_{32} \\ -\iota_1u_{12} & -\iota_2u_{32} & 1 \end{array}\right)\sim\left(\begin{array}{ccc} \cdot & \times & \circ \\ & \cdot & \bullet \\ & & \cdot \end{array}\right). \tag{50}$$

The nilpotent parameters distribution of (50) pass in (43) under exchange ι_1 and ι_2, and simultaneous reflection with respect to secondary diagonal.

Therefore, $\tilde{G}_v(2)$ is isomorphic to $G_v(2)$. Thus the permutation $\sigma = (1,3,2)$ do not lead to new contracted quantum groups.

6 Quantum groups $SO_v(4;j;\sigma)$

In this section all nonisomorphic contractions of $SO_q(4)$ are enumerated. Deformation parameter is multiplied on $J = (\sigma_1, \sigma_4) \cup (\sigma_2, \sigma_3)$, which is equal to $J = j_1 j_2 j_3$ for permutation $\sigma_0 = (1,2,3,4)$ and $J = j_1 j_3$ for $\sigma' = (1,3,4,2)$. There are not other values of J. Above-mentioned values of J correspond to nonisomorphic on the equal parameter number contracted quantum groups which have nonequivalent generating matrices for permutations σ_0 and σ'.

One-dimensional contractions. For $j_1 = \iota_1,\ J = \iota_1$ quantum Euclid group $E_v(3) = SO_v(4;\iota_1;\sigma_0)$ is obtained. For $j_2 = \iota_2$ there are two nonisomorphic quantum Newton groups: $N_v(3) = SO_v(4;\iota_2;\sigma_0),\ J = \iota_2$ and $N_z(3) = SO_z(4;\iota_2;\sigma')$ with $J = 1$. **Two-dimensional contractions.** For $j_1 = \iota_1, j_2 = \iota_2$ two nonisomorphic quantum Galilei groups: $G_v(3) = SO_v(4;\iota_1,\iota_2;\sigma_0),\ J = \iota_1\iota_2$ and $G_w(3) = SO_w(4;\iota_1,\iota_2;\sigma'),\ J = \iota_1$ are obtained. Contractions $j_1 = \iota_1, j_3 = \iota_3$ gives in result quantum groups $SO_v(4;\iota_1,\iota_3;\sigma_0)$, $J = \iota_1\iota_3$, which has not special name. Under maximal **three-dimensional contractions** $j_1 = \iota_1, j_2 = \iota_2, j_3 = \iota_3$ two nonisomorphic quantum flag groups: $F_v(4) = SO_v(4;\iota;\sigma_0),\ J = \iota_1\iota_2\iota_3$ and $F_w(4) = SO_w(4;\iota;\sigma'),\ J = \iota_1\iota_3$ are obtained.

$$E_v(3) \sim \begin{pmatrix} \cdot & \circ & \circ & \circ \\ & \cdot & \cdot & \cdot \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix}, \quad N_v(3) \sim \begin{pmatrix} \cdot & \cdot & \bullet & \bullet \\ & \cdot & \bullet & \bullet \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix},$$

$$N_z(3) \sim \begin{pmatrix} \cdot & \bullet & \bullet & \cdot \\ & \cdot & \cdot & \bullet \\ & & \cdot & \bullet \\ & & & \cdot \end{pmatrix}, \quad G_v(3) \sim \begin{pmatrix} \cdot & \circ & \times & \times \\ & \cdot & \bullet & \bullet \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix},$$

$$G_w(3) \sim \begin{pmatrix} \cdot & \times & \times & \circ \\ & \cdot & \cdot & \bullet \\ & & \cdot & \bullet \\ & & & \cdot \end{pmatrix}, \quad F_v(4) \sim \begin{pmatrix} \cdot & \circ & \times & \otimes \\ & \cdot & \bullet & \diamondsuit \\ & & \cdot & \star \\ & & & \cdot \end{pmatrix},$$

$$SO_v(4;\iota_1,\iota_3;\sigma_0) \sim \begin{pmatrix} \cdot & \circ & \circ & \triangle \\ & \cdot & \cdot & \star \\ & & \cdot & \star \\ & & & \cdot \end{pmatrix}, \quad F_w(4) \sim \begin{pmatrix} \cdot & \times & \otimes & \circ \\ & \cdot & \star & \bullet \\ & & \cdot & \diamondsuit \\ & & & \cdot \end{pmatrix},$$

where $\triangle = \iota_3,\ \star = \iota_1\iota_3,\ \diamondsuit = \iota_2\iota_3,\ \otimes = \iota_1\iota_2\iota_3$.

Thus, for quantum case there are eight different contracted groups while for classical group $SO(4)$ there are only five nonisomorphic contracted Caley–Klein groups.

7 Quantum groups $SO_v(5; j; \sigma)$

Deformation parameter is multiplied on $J = (\sigma_1, \sigma_5) \cup (\sigma_2, \sigma_4)$, which is equal to $J = j_1 j_2 j_3 j_4$ for permutation $\sigma_0 = (1,2,3,4,5)$, equal to $J = j_1 j_2 j_3$ for permutation $\sigma^1 = (1,2,5,3,4)$, equal to $J = j_1 j_2 j_4$ for permutation $\sigma^2 = (1,4,2,5,3)$, equal to $J = j_1 j_3$ for permutation $\sigma^3 = (1,3,5,4,2)$, equal to $J = j_1 j_4$ for permutation $\sigma^4 = (1,4,3,5,2)$, equal to $J = j_2 j_4$ for permutation $\sigma^5 = (2,4,1,5,3)$, equal to $J = j_1 j_3 j_4$ for permutation $\sigma^6 = (1,3,4,5,2)$, equal to $J = j_2 j_3 j_4$ for permutation $\sigma^7 = (2,3,1,4,5)$.

If contractions only on parameters j_1, j_2 are considered, then two quantum Euclid groups: $E_v(4) = SO_v(4;\iota_1;\sigma_0),\ J = \iota_1$ and $E_z(4) = SO_z(4;\iota_1;\sigma^5),\ J = 1$ with distribution of nilpotent parameters in the form

$$E_v(4) \sim \begin{pmatrix} \cdot & \circ & \circ & \circ & \circ \\ & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & & \cdot \end{pmatrix}, \qquad E_z(4) \sim \begin{pmatrix} \cdot & \circ & \cdot & \cdot & \cdot \\ & \cdot & \circ & \circ & \circ \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & & \cdot \end{pmatrix},$$

two quantum Newton groups: $N_v(4) = SO_v(4;\iota_2;\sigma_0),\ J = \iota_2$ and $N_z(4) = SO_z(4;\iota_2;\sigma^3),\ J = 1$ with generating matrices

$$N_v(4) \sim \begin{pmatrix} \cdot & \cdot & \bullet & \bullet & \bullet \\ & \cdot & \bullet & \bullet & \bullet \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & & \cdot \end{pmatrix}, \qquad N_z(4) \sim \begin{pmatrix} \cdot & \bullet & \cdot & \bullet & \bullet \\ & \cdot & \bullet & \cdot & \cdot \\ & & \cdot & \bullet & \bullet \\ & & & \cdot & \cdot \\ & & & & \cdot \end{pmatrix},$$

and two quantum Galilei groups: $G_v(4) = SO_v(4; \iota_1\iota_2; \sigma_0)$, $J = \iota_1\iota_2$ and $G_z(4) = SO_v(4; \iota_1\iota_2; \sigma^3)$, $J = \iota_1$ with generating matrices

$$G_v(4) \sim \begin{pmatrix} \cdot & \circ & \times & \times & \times \\ & \cdot & \bullet & \bullet & \bullet \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & & \cdot \end{pmatrix}, \qquad G_z(4) \sim \begin{pmatrix} \cdot & \times & \circ & \times & \times \\ & \cdot & \bullet & \cdot & \cdot \\ & & \cdot & \bullet & \bullet \\ & & & \cdot & \cdot \\ & & & & \cdot \end{pmatrix}.$$

As compared with the case $N = 3$ two quantum Newton groups are added.

In all discussed examples for $N = 3, 4, 5$ the number of nonisomorphic quantum analogues of the corresponding classical groups is equal two. It may be think that this number for any contractions do not exceed two. But this is not so. The number of nonisomorphic quantum analogues of the classical Caley–Klein groups is increased when the number of nilpotent valued contraction parameters is increased. For example, under maximal contraction $j_k = \iota_k, k = 1, \ldots, 4$ five quantum analogues of the flag group $F(5) = SO(5; \iota)$ are obtained, namely: $F_v(5) = SO_v(5; \iota; \sigma_0)$, $J = \iota_1\iota_2\iota_3\iota_4$; $F_{v_1}(5) = SO_{v_1}(5; \iota; \sigma^1)$, $J = \iota_1\iota_2\iota_3$; $F_{v_2}(5) = SO_{v_2}(5; \iota; \sigma^2)$, $J = \iota_1\iota_2\iota_4$; $F_{v_3}(5) = SO_{v_3}(5; \iota; \sigma^3)$, $J = \iota_1\iota_3$; $F_{v_4}(5) = SO_{v_4}(5; \iota; \sigma^4)$, $J = \iota_1\iota_4$. All they have generating matrices with nonequivalent distributions of nilpotent parameters.

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Appendix 1

R-matrix of quantum group $SO_q(N)$ in Cartesian basis

$$\tilde{R}_q = (D \otimes D)R(D \otimes D)^{-1} =$$

$$= I + \frac{1}{2}(q-1)(1-q^{-1}) \sum_{\substack{k=1 \\ k \neq k'}}^{N} (e_{kk} \otimes e_{kk} + e_{kk} \otimes e_{k'k'}) + \frac{\lambda}{2} \sum_{\substack{k=1 \\ k \neq k'}}^{N} (e_{k'k} \otimes e_{kk'} - e_{k'k} \otimes e_{k'k})$$

$$+ \frac{\lambda}{2} \sum_{k=1}^{n} (e_{k',n+1} \otimes e_{n+1,k'} - ie_{k',n+1} \otimes e_{n+1,k} + ie_{k,n+1} \otimes e_{n+1,k'} + e_{k,n+1} \otimes e_{n+1,k}$$

$$+e_{n+1,k}\otimes e_{k,n+1}+ie_{n+1,k}\otimes e_{k',n+1}-ie_{n+1,k'}\otimes e_{k,n+1}+e_{n+1,k'}\otimes e_{k',n+1})$$

$$-\frac{\lambda}{2}\sum_{k=1}^{n}q^{-\rho_k}(-ie_{k',n+1}\otimes e_{k,n+1}+e_{k',n+1}\otimes e_{k',n+1}+e_{k,n+1}\otimes e_{k,n+1}+ie_{k,n+1}\otimes e_{k',n+1}$$

$$+ie_{n+1,k}\otimes e_{n+1,k'}+e_{n+1,k}\otimes e_{n+1,k}+e_{n+1,k'}\otimes e_{n+1,k'}-ie_{n+1,k'}\otimes e_{n+1,k})$$

$$+\frac{\lambda}{4}\sum_{\substack{k,p=1\\ k>p,\ k,p\neq n+1}}^{N}(e_{kp}\otimes e_{pk}+e_{kp}\otimes e_{p'k'}+ie_{kp}\otimes e_{p'k}-ie_{kp}\otimes e_{pk'}$$

$$+e_{k'p'}\otimes e_{pk}+e_{k'p'}\otimes e_{p'k'}+ie_{k'p'}\otimes e_{p'k}-ie_{k'p'}\otimes e_{pk'}$$

$$+ie_{k'p}\otimes e_{pk}+ie_{k'p}\otimes e_{p'k'}-e_{k'p}\otimes e_{p'k}+e_{k'p}\otimes e_{pk'}$$

$$-ie_{kp'}\otimes e_{pk}-ie_{kp'}\otimes e_{p'k'}+e_{kp'}\otimes e_{p'k}-e_{kp'}\otimes e_{pk'})$$

$$-\frac{\lambda}{4}\sum_{\substack{k,p=1\\ k>p,\ k,p\neq n+1}}^{N}q^{\rho_k-\rho_p}(e_{kp}\otimes e_{k'p'}+e_{kp}\otimes e_{kp}+ie_{kp}\otimes e_{kp'}-ie_{kp}\otimes e_{k'p}$$

$$+e_{k'p'}\otimes e_{k'p'}+e_{k'p'}\otimes e_{kp}+ie_{k'p'}\otimes e_{kp'}-ie_{k'p'}\otimes e_{k'p}$$

$$+ie_{k'p}\otimes e_{k'p'}+ie_{k'p}\otimes e_{kp}-e_{k'p}\otimes e_{kp'}+e_{k'p}\otimes e_{k'p}$$

$$-ie_{kp'}\otimes e_{k'p'}-ie_{kp'}\otimes e_{kp}+e_{kp'}\otimes e_{kp'}-e_{kp'}\otimes e_{k'p}),\quad \lambda=q-q^{-1}.$$

Appendix 2

Antipode of quantum group $SO_v(N;j;\sigma)$ in Cartesian basis

Antipode of Cartesian generators of quantum group $SO_v(N;j;\sigma)$, $N=2n+1$ is obtained by formula

$$S(U(j;\sigma))=\tilde{C}_v(j)U^t(j;\sigma)\tilde{C}_v^{-1}(j)$$

with the help of matrix $\tilde{C}_v(j)=D^{-1}C_v(j)(D^t)^{-1}$ in the following form

$$S(u_{\sigma_k\sigma_{n+1}})=u_{\sigma_{n+1}\sigma_k}\cosh(Jv\rho_k)+iu_{\sigma_{n+1}\sigma_{k'}}\frac{(\sigma_{k'},\sigma_{n+1})}{(\sigma_k,\sigma_{n+1})}\sinh(Jv\rho_k),$$

$$S(u_{\sigma_{n+1}\sigma_k})=u_{\sigma_k\sigma_{n+1}}\cosh(Jv\rho_k)+iu_{\sigma_{k'}\sigma_{n+1}}\frac{(\sigma_{k'},\sigma_{n+1})}{(\sigma_k,\sigma_{n+1})}\sinh(Jv\rho_k),$$

$$S(u_{\sigma_{n+1+k}\sigma_{n+1}}) = u_{\sigma_{n+1}\sigma_{n+1+k}} \cosh(Jv\rho_{n+1-k}) -$$

$$-iu_{\sigma_{n+1}\sigma_{n+1-k}} \frac{(\sigma_{n+1-k}, \sigma_{n+1})}{(\sigma_{n+1+k}, \sigma_{n+1})} \sinh(Jv\rho_{n+1-k}),$$

$$S(u_{\sigma_{n+1}\sigma_{n+1+k}}) = u_{\sigma_{n+1+k}\sigma_{n+1}} \cosh(Jv\rho_{n+1-k}) -$$

$$-iu_{\sigma_{n+1-k}\sigma_{n+1}} \frac{(\sigma_{n+1-k}, \sigma_{n+1})}{(\sigma_{n+1+k}, \sigma_{n+1})} \sinh(Jv\rho_{n+1-k}),$$

$$S(u_{\sigma_k\sigma_p}) = u_{\sigma_p\sigma_k} \cosh(Jv\rho_k) \cosh(Jv\rho_p) -$$

$$-u_{\sigma_{p'}\sigma_{k'}} \frac{(\sigma_{k'}, \sigma_{p'})}{(\sigma_k, \sigma_p)} \sinh(Jv\rho_k) \sinh(Jv\rho_p) +$$

$$+i \left(u_{\sigma_p\sigma_{k'}} \frac{(\sigma_{k'}, \sigma_p)}{(\sigma_k, \sigma_p)} \sinh(Jv\rho_k) \cosh(Jv\rho_p) + \right.$$

$$\left. + u_{\sigma_{p'}\sigma_k} \frac{(\sigma_k, \sigma_{p'})}{(\sigma_k, \sigma_p)} \cosh(Jv\rho_k) \sinh(Jv\rho_p) \right),$$

$$S(u_{\sigma_k\sigma_{n+1+p}}) = u_{\sigma_{n+1+p}\sigma_k} \cosh(Jv\rho_k) \cosh(Jv\rho_{n+1-p}) +$$

$$+u_{\sigma_{n+1-p}\sigma_{k'}} \frac{(\sigma_{k'}, \sigma_{n+1-p})}{(\sigma_k, \sigma_{n+1+p})} \sinh(Jv\rho_k) \sinh(Jv\rho_{n+1-p}) +$$

$$+i \left(u_{\sigma_{n+1+p}\sigma_{k'}} \frac{(\sigma_{k'}, \sigma_{n+1+p})}{(\sigma_k, \sigma_{n+1+p})} \sinh(Jv\rho_k) \cosh(Jv\rho_{n+1-p}) - \right.$$

$$\left. -u_{\sigma_{n+1-p}\sigma_k} \frac{(\sigma_k, \sigma_{n+1-p})}{(\sigma_k, \sigma_{n+1+p})} \cosh(Jv\rho_k) \sinh(Jv\rho_{n+1-p}) \right),$$

$$S(u_{\sigma_{n+1+k}\sigma_p}) = u_{\sigma_p\sigma_{n+1+k}} \cosh(Jv\rho_{n+1-k}) \cosh(Jv\rho_p) +$$

$$+u_{\sigma_{p'}\sigma_{n+1-k}} \frac{(\sigma_{n+1-k}, \sigma_{p'})}{(\sigma_{n+1+k}, \sigma_p)} \sinh(Jv\rho_{n+1-k}) \sinh(Jv\rho_p) +$$

$$+i \left(u_{\sigma_{p'}\sigma_{n+1+k}} \frac{(\sigma_{n+1+k}, \sigma_{p'})}{(\sigma_{n+1+k}, \sigma_p)} \cosh(Jv\rho_{n+1-k}) \sinh(Jv\rho_p) - \right.$$

$$\left. -u_{\sigma_p\sigma_{n+1-k}} \frac{(\sigma_{n+1-k}, \sigma_p)}{(\sigma_{n+1+k}, \sigma_p)} \sinh(Jv\rho_{n+1-k}) \cosh(Jv\rho_p) \right),$$

$$S(u_{\sigma_{n+1+k}\sigma_{n+1+p}}) = u_{\sigma_{n+1+p}\sigma_{n+1+k}} \cosh(Jv\rho_{n+1-k}) \cosh(Jv\rho_{n+1-p}) -$$

$$-u_{\sigma_{n+1-p}\sigma_{n+1-k}} \frac{(\sigma_{n+1-k}, \sigma_{n+1-p})}{(\sigma_{n+1+k}, \sigma_{n+1+p})} \sinh(Jv\rho_{n+1-k}) \sinh(Jv\rho_{n+1-p}) -$$

$$-i\left(u_{\sigma_{n+1-p}\sigma_{n+1+k}}\frac{(\sigma_{n+1+k},\sigma_{n+1-p})}{(\sigma_{n+1+k},\sigma_{n+1+p})}\cosh(Jv\rho_{n+1-k})\sinh(Jv\rho_{n+1-p})+\right.$$

$$\left.+u_{\sigma_{n+1+p}\sigma_{n+1-k}}\frac{(\sigma_{n+1-k},\sigma_{n+1+p})}{(\sigma_{n+1+k},\sigma_{n+1+p})}\sinh(Jv\rho_{n+1-k})\cosh(Jv\rho_{n+1-p})\right), \qquad (51)$$

where $k,p=1,\ldots,n$. Antipode of quantum group $SO_v(N;j;\sigma),\ N=2n$ is given by above-mentioned formulae with the replacement $n+1$ on n.

Appendix 3

(v,j)-orthogonality relations of quantum group $SO_v(N;\sigma;j)$ in Cartesian basis

Additional relations $U(j;\sigma)\tilde{C}_v(j)U^t(j;\sigma)=\tilde{C}_v(j)$, where $\tilde{C}_v(j)=D^{-1}C_v(j)(D^t)^{-1}$ are in the form

$$u_{\sigma_k\sigma_{n+1}}u_{\sigma_p\sigma_{n+1}}(\sigma_k,\sigma_{n+1})(\sigma_p,\sigma_{n+1})+$$

$$+\sum_{s=1}^{n}\Big\{u_{\sigma_k\sigma_s}u_{\sigma_p\sigma_s}(\sigma_k,\sigma_s)(\sigma_p,\sigma_s)\cosh(Jv\rho_s)+$$

$$+u_{\sigma_k\sigma_{n+1+s}}u_{\sigma_p\sigma_{n+1+s}}(\sigma_k,\sigma_{n+1+s})(\sigma_p,\sigma_{n+1+s})\cosh(Jv\rho_{n+1-s})+$$

$$+i\Big[u_{\sigma_k\sigma_{n+1-s}}u_{\sigma_p\sigma_{n+1+s}}(\sigma_k,\sigma_{n+1-s})(\sigma_p,\sigma_{n+1+s})\sinh(Jv\rho_{n+1-s})-$$

$$-u_{\sigma_k\sigma_{s'}}u_{\sigma_p\sigma_s}(\sigma_k,\sigma_{s'})(\sigma_p,\sigma_s)\sinh(Jv\rho_s)\Big]\Big\}=\delta_{kp}\cosh(Jv\rho_k), \qquad (52)$$

$$u_{\sigma_{n+1+k}\sigma_{n+1}}u_{\sigma_{n+1+p}\sigma_{n+1}}(\sigma_{n+1+k},\sigma_{n+1})(\sigma_{n+1+p},\sigma_{n+1})+$$

$$+\sum_{s=1}^{n}\Big\{u_{\sigma_{n+1+k}\sigma_s}u_{\sigma_{n+1+p}\sigma_s}(\sigma_{n+1+k},\sigma_s)(\sigma_{n+1+p},\sigma_s)\cosh(Jv\rho_s)+$$

$$+u_{\sigma_{n+1+k}\sigma_{n+1+s}}u_{\sigma_{n+1+p}\sigma_{n+1+s}}(\sigma_{n+1+k},\sigma_{n+1+s})(\sigma_{n+1+p},\sigma_{n+1+s})\cosh(Jv\rho_{n+1-s})+$$

$$+i\Big[u_{\sigma_{n+1+k}\sigma_{n+1-s}}u_{\sigma_{n+1+p}\sigma_{n+1+s}}(\sigma_{n+1+k},\sigma_{n+1-s})(\sigma_{n+1+p},\sigma_{n+1+s})\sinh(Jv\rho_{n+1-s})-$$

$$-u_{\sigma_{n+1+k}\sigma_{s'}}u_{\sigma_{n+1+p}\sigma_s}(\sigma_{n+1+k},\sigma_{s'})(\sigma_{n+1+p},\sigma_s)\sinh(Jv\rho_s)\Big]\Big\}=$$

$$=\delta_{kp}\cosh(Jv\rho_{n+1-k}), \qquad (53)$$

$$u_{\sigma_k\sigma_{n+1}}u_{\sigma_{n+1+p}\sigma_{n+1}}(\sigma_k,\sigma_{n+1})(\sigma_{n+1+p},\sigma_{n+1})+$$

$$+\sum_{s=1}^{n}\Big\{u_{\sigma_k\sigma_s}u_{\sigma_{n+1+p}\sigma_s}(\sigma_k,\sigma_s)(\sigma_{n+1+p},\sigma_s)\cosh(Jv\rho_s)+$$

$$+u_{\sigma_k\sigma_{n+1+s}}u_{\sigma_{n+1+p}\sigma_{n+1+s}}(\sigma_k,\sigma_{n+1+s})(\sigma_{n+1+p},\sigma_{n+1+s})\cosh(Jv\rho_{n+1-s})+$$

$$+i\Big[u_{\sigma_k\sigma_{n+1-s}}u_{\sigma_{n+1+p}\sigma_{n+1+s}}(\sigma_k,\sigma_{n+1-s})(\sigma_{n+1+p},\sigma_{n+1+s})\sinh(Jv\rho_{n+1-s})-$$

$$-u_{\sigma_k\sigma_{s'}}u_{\sigma_{n+1+p}\sigma_s}(\sigma_k,\sigma_{s'})(\sigma_{n+1+p},\sigma_s)\sinh(Jv\rho_s)\Big]\Big\}=i\delta_{n+1-k,p}\sinh(Jv\rho_k), \tag{54}$$

$$u_{\sigma_{n+1+k}\sigma_{n+1}}u_{\sigma_p\sigma_{n+1}}(\sigma_{n+1+k},\sigma_{n+1})(\sigma_p,\sigma_{n+1})+$$

$$+\sum_{s=1}^{n}\Big\{u_{\sigma_{n+1+k}\sigma_s}u_{\sigma_p\sigma_s}(\sigma_{n+1+k},\sigma_s)(\sigma_p,\sigma_s)\cosh(Jv\rho_s)+$$

$$+u_{\sigma_{n+1+k}\sigma_{n+1+s}}u_{\sigma_p\sigma_{n+1+s}}(\sigma_{n+1+k},\sigma_{n+1+s})(\sigma_p,\sigma_{n+1+s})\cosh(Jv\rho_{n+1-s})+$$

$$+i\Big[u_{\sigma_{n+1+k}\sigma_{n+1-s}}u_{\sigma_p\sigma_{n+1+s}}(\sigma_{n+1+k},\sigma_{n+1-s})(\sigma_p,\sigma_{n+1+s})\sinh(Jv\rho_{n+1-s})-$$

$$-u_{\sigma_{n+1+k}\sigma_{s'}}u_{\sigma_p\sigma_s}(\sigma_{n+1+k},\sigma_{s'})(\sigma_p,\sigma_s)\sinh(Jv\rho_s)\Big]\Big\}=-i\delta_{n+1-k,p}\sinh(Jv\rho_p), \tag{55}$$

$$u^2_{\sigma_{n+1}\sigma_{n+1}}+\sum_{k=1}^{n}\Big\{u^2_{\sigma_{n+1}\sigma_k}(\sigma_{n+1},\sigma_k)^2\cosh(Jv\rho_k)+$$

$$+u^2_{\sigma_{n+1}\sigma_{n+1+k}}(\sigma_{n+1},\sigma_{n+1+k})^2\cosh(Jv\rho_{n+1-k})+$$

$$+i\Big[u_{\sigma_{n+1}\sigma_k}u_{\sigma_{n+1}\sigma_{k'}}(\sigma_{n+1},\sigma_k)(\sigma_{n+1},\sigma_{k'})\sinh(Jv\rho_k)-$$

$$-u_{\sigma_{n+1}\sigma_{n+1+k}}u_{\sigma_{n+1}\sigma_{n+1-k}}(\sigma_{n+1},\sigma_{n+1+k})(\sigma_{n+1},\sigma_{n+1-k})\sinh(Jv\rho_{n+1-k})\Big]\Big\}=1, \tag{56}$$

$$u_{\sigma_k\sigma_{n+1}}u_{\sigma_{n+1}\sigma_{n+1}}(\sigma_k,\sigma_{n+1})+\sum_{s=1}^{n}\Big\{u_{\sigma_k\sigma_s}u_{\sigma_{n+1}\sigma_s}(\sigma_k,\sigma_s)(\sigma_{n+1},\sigma_s)\cosh(Jv\rho_s)+$$

$$+u_{\sigma_k\sigma_{n+1+s}}u_{\sigma_{n+1}\sigma_{n+1+s}}(\sigma_k,\sigma_{n+1+s})(\sigma_{n+1},\sigma_{n+1+s})\cosh(Jv\rho_{n+1-s})+$$

$$+i\Big[u_{\sigma_k\sigma_s}u_{\sigma_{n+1}\sigma_{s'}}(\sigma_k,\sigma_s)(\sigma_{n+1},\sigma_{s'})\sinh(Jv\rho_s)-$$

$$-u_{\sigma_k\sigma_{n+1+s}}u_{\sigma_{n+1}\sigma_{n+1-s}}(\sigma_k,\sigma_{n+1+s})(\sigma_{n+1},\sigma_{n+1-s})\sinh(Jv\rho_{n+1-s})\Big]\Big\}=0, \tag{57}$$

$$u_{\sigma_{n+1}\sigma_{n+1}}u_{\sigma_k\sigma_{n+1}}(\sigma_k,\sigma_{n+1})+\sum_{s=1}^{n}\Big\{u_{\sigma_{n+1}\sigma_s}u_{\sigma_k\sigma_s}(\sigma_{n+1},\sigma_s)(\sigma_k,\sigma_s)\cosh(Jv\rho_s)+$$

$$+u_{\sigma_{n+1}\sigma_{n+1+s}}u_{\sigma_k\sigma_{n+1+s}}(\sigma_{n+1},\sigma_{n+1+s})(\sigma_k,\sigma_{n+1+s})\cosh(Jv\rho_{n+1-s})-$$

$$-i\Big[u_{\sigma_{n+1}\sigma_{s'}}u_{\sigma_k\sigma_s}(\sigma_{n+1},\sigma_{s'})(\sigma_k,\sigma_s)\sinh(Jv\rho_s)-$$

$$-u_{\sigma_{n+1}\sigma_{n+1-s}}u_{\sigma_k\sigma_{n+1+s}}(\sigma_{n+1},\sigma_{n+1-s})(\sigma_k,\sigma_{n+1+s})\sinh(Jv\rho_{n+1-s})\Big]\Big\}=0, \tag{58}$$

$$u_{\sigma_{n+1}\sigma_{n+1}}u_{\sigma_{n+1+k}\sigma_{n+1}}(\sigma_{n+1+k},\sigma_{n+1})+$$
$$+\sum_{s=1}^{n}\Big\{u_{\sigma_{n+1}\sigma_s}u_{\sigma_{n+1+k}\sigma_s}(\sigma_{n+1},\sigma_s)(\sigma_{n+1+k},\sigma_s)\cosh(Jv\rho_s)+$$
$$+u_{\sigma_{n+1}\sigma_{n+1+s}}u_{\sigma_{n+1+k}\sigma_{n+1+s}}(\sigma_{n+1},\sigma_{n+1+s})(\sigma_{n+1+k},\sigma_{n+1+s})\cosh(Jv\rho_{n+1-s})-$$
$$-i\Big[u_{\sigma_{n+1}\sigma_{s'}}u_{\sigma_{n+1+k}\sigma_s}(\sigma_{n+1+k},\sigma_s)(\sigma_{n+1},\sigma_{s'})\sinh(Jv\rho_s)-$$
$$-u_{\sigma_{n+1}\sigma_{n+1-s}}u_{\sigma_{n+1+k}\sigma_{n+1+s}}(\sigma_{n+1+k},\sigma_{n+1+s})(\sigma_{n+1},\sigma_{n+1-s})\sinh(Jv\rho_{n+1-s})\Big]\Big\}=0, \tag{59}$$

$$u_{\sigma_{n+1+k}\sigma_{n+1}}u_{\sigma_{n+1}\sigma_{n+1}}(\sigma_{n+1+k},\sigma_{n+1})+$$
$$+\sum_{s=1}^{n}\Big\{u_{\sigma_{n+1+k}\sigma_s}u_{\sigma_{n+1}\sigma_s}(\sigma_{n+1},\sigma_s)(\sigma_{n+1+k},\sigma_s)\cosh(Jv\rho_s)+$$
$$+u_{\sigma_{n+1+k}\sigma_{n+1+s}}u_{\sigma_{n+1}\sigma_{n+1+s}}(\sigma_{n+1+k},\sigma_{n+1+s})(\sigma_{n+1},\sigma_{n+1+s})\cosh(Jv\rho_{n+1-s})+$$
$$+i\Big[u_{\sigma_{n+1+k}\sigma_s}u_{\sigma_{n+1}\sigma_{s'}}(\sigma_{n+1+k},\sigma_s)(\sigma_{n+1},\sigma_{s'})\sinh(Jv\rho_s)-$$
$$-u_{\sigma_{n+1+k}\sigma_{n+1+s}}u_{\sigma_{n+1}\sigma_{n+1-s}}(\sigma_{n+1+k},\sigma_{n+1+s})(\sigma_{n+1},\sigma_{n+1-s})\sinh(Jv\rho_{n+1-s})\Big]\Big\}=0, \tag{60}$$

and additional relations $U^t(j;\sigma)\tilde{C}_v^{-1}(j)U(j;\sigma)=\tilde{C}_v^{-1}(j)$ are equal

$$u_{\sigma_{n+1}\sigma_k}u_{\sigma_{n+1}\sigma_p}(\sigma_{n+1},\sigma_k)(\sigma_{n+1},\sigma_p)+\sum_{s=1}^{n}\Big\{u_{\sigma_s\sigma_k}u_{\sigma_s\sigma_p}(\sigma_s,\sigma_k)(\sigma_s,\sigma_p)\cosh(Jv\rho_s)+$$
$$+u_{\sigma_{n+1+s}\sigma_k}u_{\sigma_{n+1+s}\sigma_p}(\sigma_{n+1+s},\sigma_k)(\sigma_{n+1+s},\sigma_p)\cosh(Jv\rho_{n+1-s})+$$
$$+i\Big[u_{\sigma_{s'}\sigma_k}u_{\sigma_s\sigma_p}(\sigma_{s'},\sigma_k)(\sigma_s,\sigma_p)\sinh(Jv\rho_s)-$$
$$-u_{\sigma_{n+1-s}\sigma_k}u_{\sigma_{n+1+s}\sigma_p}(\sigma_{n+1-s},\sigma_k)(\sigma_{n+1+s},\sigma_p)\sinh(Jv\rho_{n+1-s})\Big]\Big\}=\delta_{kp}\cosh(Jv\rho_k),$$

$$u_{\sigma_{n+1}\sigma_{n+1+k}}u_{\sigma_{n+1}\sigma_{n+1+p}}(\sigma_{n+1},\sigma_{n+1+k})(\sigma_{n+1},\sigma_{n+1+p})+$$
$$+\sum_{s=1}^{n}\Big\{u_{\sigma_s\sigma_{n+1+k}}u_{\sigma_s\sigma_{n+1+p}}(\sigma_s,\sigma_{n+1+k})(\sigma_s,\sigma_{n+1+p})\cosh(Jv\rho_s)+$$
$$+u_{\sigma_{n+1+s}\sigma_{n+1+k}}u_{\sigma_{n+1+s}\sigma_{n+1+p}}(\sigma_{n+1+s},\sigma_{n+1+k})(\sigma_{n+1+s},\sigma_{n+1+p})\cosh(Jv\rho_{n+1-s})+$$
$$+i\Big[u_{\sigma_{s'}\sigma_{n+1+k}}u_{\sigma_s\sigma_{n+1+p}}(\sigma_{s'},\sigma_{n+1+k})(\sigma_s,\sigma_{n+1+p})\sinh(Jv\rho_s)-$$

$$-u_{\sigma_{n+1-s}\sigma_{n+1+k}}u_{\sigma_{n+1+s}\sigma_{n+1+p}}(\sigma_{n+1-s},\sigma_{n+1+k})(\sigma_{n+1+s},\sigma_{n+1+p})\sinh(Jv\rho_{n+1-s})\Big]\Big\} =$$
$$= \delta_{kp}\cosh(Jv\rho_{n+1-k}),$$
$$u_{\sigma_{n+1}\sigma_k}u_{\sigma_{n+1}\sigma_{n+1+p}}(\sigma_{n+1},\sigma_k)(\sigma_{n+1},\sigma_{n+1+p})+$$
$$+\sum_{s=1}^{n}\Big\{u_{\sigma_s\sigma_k}u_{\sigma_s\sigma_{n+1+p}}(\sigma_s,\sigma_k)(\sigma_s,\sigma_{n+1+p})\cosh(Jv\rho_s)+$$
$$+u_{\sigma_{n+1+s}\sigma_k}u_{\sigma_{n+1+s}\sigma_{n+1+p}}(\sigma_{n+1+s},\sigma_k)(\sigma_{n+1+s},\sigma_{n+1+p})\cosh(Jv\rho_{n+1-s})+$$
$$+i\Big[u_{\sigma_{s'}\sigma_k}u_{\sigma_s\sigma_{n+1+p}}(\sigma_{s'},\sigma_k)(\sigma_s,\sigma_{n+1+p})\sinh(Jv\rho_s)-$$
$$-u_{\sigma_{n+1-s}\sigma_k}u_{\sigma_{n+1+s}\sigma_{n+1+p}}(\sigma_{n+1-s},\sigma_k)(\sigma_{n+1+s},\sigma_{n+1+p})\sinh(Jv\rho_{n+1-s})\Big]\Big\} =$$
$$= -i\delta_{n+1-k,p}\sinh(Jv\rho_k),$$
$$u_{\sigma_{n+1}\sigma_{n+1+k}}u_{\sigma_{n+1}\sigma_p}(\sigma_{n+1},\sigma_{n+1+k})(\sigma_{n+1},\sigma_p)+$$
$$+\sum_{s=1}^{n}\Big\{u_{\sigma_s\sigma_{n+1+k}}u_{\sigma_s\sigma_p}(\sigma_s,\sigma_{n+1+k})(\sigma_s,\sigma_p)\cosh(Jv\rho_s)+$$
$$+u_{\sigma_{n+1+s}\sigma_{n+1+k}}u_{\sigma_{n+1+s}\sigma_p}(\sigma_{n+1+s},\sigma_{n+1+k})(\sigma_{n+1+s},\sigma_p)\cosh(Jv\rho_{n+1-s})+$$
$$+i\Big[u_{\sigma_{s'}\sigma_{n+1+k}}u_{\sigma_s\sigma_p}(\sigma_{s'},\sigma_{n+1+k})(\sigma_s,\sigma_p)\sinh(Jv\rho_s)-$$
$$-u_{\sigma_{n+1-s}\sigma_{n+1+k}}u_{\sigma_{n+1+s}\sigma_p}(\sigma_{n+1-s},\sigma_{n+1+k})(\sigma_{n+1+s},\sigma_p)\sinh(Jv\rho_{n+1-s})\Big]\Big\} =$$
$$= i\delta_{n+1-k,p}\sinh(Jv\rho_p),$$
$$u^2_{\sigma_{n+1}\sigma_{n+1}}+\sum_{k=1}^{n}\Big\{u^2_{\sigma_k\sigma_{n+1}}(\sigma_k,\sigma_{n+1})^2\cosh(Jv\rho_k)+$$
$$+u^2_{\sigma_{n+1+k}\sigma_{n+1}}(\sigma_{n+1+k},\sigma_{n+1})^2\cosh(Jv\rho_{n+1-k})+$$
$$+i\Big[u_{\sigma_{n+1+k}\sigma_{n+1}}u_{\sigma_{n+1-k}\sigma_{n+1}}(\sigma_{n+1+k},\sigma_{n+1})(\sigma_{n+1-k},\sigma_{n+1})\sinh(Jv\rho_{n+1-k})-$$
$$-u_{\sigma_k\sigma_{n+1}}u_{\sigma_{k'}\sigma_{n+1}}(\sigma_k,\sigma_{n+1})(\sigma_{k'},\sigma_{n+1})\sinh(Jv\rho_k)\Big]\Big\} = 1,$$
$$u_{\sigma_{n+1}\sigma_k}u_{\sigma_{n+1}\sigma_{n+1}}(\sigma_{n+1},\sigma_k)+\sum_{p=1}^{n}\Big\{u_{\sigma_p\sigma_k}u_{\sigma_p\sigma_{n+1}}(\sigma_p,\sigma_k)(\sigma_p,\sigma_{n+1})\cosh(Jv\rho_p)+$$
$$+u_{\sigma_{n+1+p}\sigma_k}u_{\sigma_{n+1+p}\sigma_{n+1}}(\sigma_{n+1+p},\sigma_k)(\sigma_{n+1+p},\sigma_{n+1})\cosh(Jv\rho_{n+1-p})-$$
$$-i\Big[u_{\sigma_p\sigma_k}u_{\sigma_{p'}\sigma_{n+1}}(\sigma_p,\sigma_k)(\sigma_{p'},\sigma_{n+1})\sinh(Jv\rho_p)-$$
$$-u_{\sigma_{n+1+p}\sigma_k}u_{\sigma_{n+1-p}\sigma_{n+1}}(\sigma_{n+1+p},\sigma_k)(\sigma_{n+1-p},\sigma_{n+1})\sinh(Jv\rho_{n+1-p})\Big]\Big\} = 0,$$

$$
\begin{gathered}
u_{\sigma_{n+1}\sigma_{n+1}}u_{\sigma_{n+1}\sigma_k}(\sigma_{n+1},\sigma_k)+ \\
+\sum_{p=1}^{n}\Big\{u_{\sigma_p\sigma_{n+1}}u_{\sigma_p\sigma_k}(\sigma_p,\sigma_{n+1})(\sigma_p,\sigma_k)\cosh(Jv\rho_p)+ \\
+u_{\sigma_{n+1+p}\sigma_{n+1}}u_{\sigma_{n+1+p}\sigma_k}(\sigma_{n+1+p},\sigma_{n+1})(\sigma_{n+1+p},\sigma_k)\cosh(Jv\rho_{n+1-p})+ \\
+i\Big[u_{\sigma_{p'}\sigma_{n+1}}u_{\sigma_p\sigma_k}(\sigma_{p'},\sigma_{n+1})(\sigma_p,\sigma_k)\sinh(Jv\rho_p)- \\
-u_{\sigma_{n+1+p}\sigma_{n+1}}u_{\sigma_{n+1+p}\sigma_k}(\sigma_{n+1+p},\sigma_{n+1})(\sigma_{n+1+p},\sigma_k)\sinh(Jv\rho_{n+1-p})\Big]\Big\}=0, \\
u_{\sigma_{n+1}\sigma_{n+1}}u_{\sigma_{n+1}\sigma_{n+1+k}}(\sigma_{n+1},\sigma_{n+1+k})+ \\
+\sum_{p=1}^{n}\Big\{u_{\sigma_p\sigma_{n+1}}u_{\sigma_p\sigma_{n+1+k}}(\sigma_p,\sigma_{n+1})(\sigma_p,\sigma_{n+1+k})\cosh(Jv\rho_p)+ \\
+u_{\sigma_{n+1+p}\sigma_{n+1}}u_{\sigma_{n+1+p}\sigma_{n+1+k}}(\sigma_{n+1+p},\sigma_{n+1})(\sigma_{n+1+p},\sigma_{n+1+k})\cosh(Jv\rho_{n+1-p})+ \\
+i\Big[u_{\sigma_{p'}\sigma_{n+1}}u_{\sigma_p\sigma_{n+1+k}}(\sigma_{p'},\sigma_{n+1})(\sigma_p,\sigma_{n+1+k})\sinh(Jv\rho_p)- \\
-u_{\sigma_{n+1-p}\sigma_{n+1}}u_{\sigma_{n+1+p}\sigma_{n+1+k}}(\sigma_{n+1-p},\sigma_{n+1})(\sigma_{n+1+p},\sigma_{n+1+k})\sinh(Jv\rho_{n+1-p})\Big]\Big\}=0, \\
u_{\sigma_{n+1}\sigma_{n+1+k}}u_{\sigma_{n+1}\sigma_{n+1}}(\sigma_{n+1},\sigma_{n+1+k})+ \\
+\sum_{p=1}^{n}\Big\{u_{\sigma_p\sigma_{n+1+k}}u_{\sigma_p\sigma_{n+1}}(\sigma_p,\sigma_{n+1+k})(\sigma_p,\sigma_{n+1})\cosh(Jv\rho_p)+ \\
+u_{\sigma_{n+1+p}\sigma_{n+1+k}}u_{\sigma_{n+1+p}\sigma_{n+1}}(\sigma_{n+1+p},\sigma_{n+1+k})(\sigma_{n+1+p},\sigma_{n+1})\cosh(Jv\rho_{n+1-p})- \\
-i\Big[u_{\sigma_p\sigma_{n+1+k}}u_{\sigma_{p'}\sigma_{n+1}}(\sigma_p,\sigma_{n+1+k})(\sigma_{p'},\sigma_{n+1})\sinh(Jv\rho_p)- \\
-u_{\sigma_{n+1+p}\sigma_{n+1+k}}u_{\sigma_{n+1-p}\sigma_{n+1}}(\sigma_{n+1+p},\sigma_{n+1+k})(\sigma_{n+1-p},\sigma_{n+1})\sinh(Jv\rho_{n+1-p})\Big]\Big\}=0,
\end{gathered}
\tag{61}
$$

where $k,p=1,\ldots,n$. (v,j)-orthogonality relations of quantum group $SO_v(N;j;\sigma)$, $N=2n$ are given by above-mentioned formulae with the replacement $n+1$ on n.

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